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LAYER HEAT POTENTIALS FOR A BOUNDED CYLINDER WITH FRACTAL LATERAL BOUNDARY

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1. Introduction

Let D be a bounded smooth domain in \mathbf{R}^d and set

$$\Omega_D = D \times (0, T) \quad \text{and} \quad S_D = \partial D \times [0, T].$$

The double layer heat potential Φf of $f \in L^p(S_D)$ is defined by

$$(1.1) \quad \Phi f(X) = - \int_0^T \int_{\partial D} \langle \nabla_y W(X - Y), n_y \rangle f(Y) d\sigma(y) ds$$

for $X = (x, t) \in (R^d \setminus \partial D) \times \mathbf{R}$, where $\langle \cdot, \cdot \rangle$ is the inner product in \mathbf{R}^d , n_y is the unit outer normal to ∂D , σ is the surface measure on ∂D and W is the fundamental solution for the heat operator, i.e.,

$$W(X) = W(x, t) = \begin{cases} \frac{\exp\left(-\frac{|x|^2}{4t}\right)}{(4\pi t)^{d/2}} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The double layer heat potential is important not only physically but also mathematically. For example, R. M. Brown proved that the solution to the initial-Dirichlet problem in a Lipschitz cylinder for the heat operator can be written by a double layer heat potential and the solution to the initial-Neumann problem in a Lipschitz cylinder for the heat operator is given by a single layer heat potential (cf. $[B_1]$, $[B_2]$).

If D is a bounded domain with fractal boundary, then n_y and the surface measure can not be defined. But if D has a smooth boundary and f is a C^1 -function on \mathbf{R}^{d+1} with compact support, then we see by the Green formula that for $X = (x, t) \in D \times \mathbf{R}$

$$(1.2) \quad \begin{aligned} \Phi f(X) = & \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y f(Y), \nabla_y W(X - Y) \rangle dy \\ & + \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} f(Y) \Delta_y W(X - Y) dy \end{aligned}$$

and for $X = (x, t) \in (\mathbf{R}^d \setminus \overline{D}) \times \mathbf{R}$

$$(1.3) \quad \begin{aligned} \Phi f(x) = & - \int_0^T ds \int_D \langle \nabla_y f(Y), \nabla_y W(X - Y) \rangle dy \\ & - \int_0^T ds \int_D f(Y) \Delta_y W(X - Y) dy. \end{aligned}$$

So we see that, if a function f defined on S_D can be extended to be a function $\mathcal{E}(f)$ on $\mathbf{R}^d \times [0, T]$ such that for each $t \in [0, T]$ the function $x \mapsto \mathcal{E}(f)(x, t)$ is a C^1 -function on $\mathbf{R}^d \setminus \partial D$ and, for each $x \in \mathbf{R}^d \setminus \partial D$ and each j ($j = 1, 2, \dots, d$) the function $t \mapsto \frac{\partial \mathcal{E}(f)}{\partial x_j}(x, t)$ is measurable, then the right-hand sides of (1.2) and (1.3) may be defined.

In this paper we assume that D is a bounded domain in \mathbf{R}^d ($d \geq 2$) and ∂D is a β -set satisfying $d - 1 \leq \beta < d$. Here, according to [JW] we say that a closed set F is a β -set if there exist a positive Radon measure μ on F and positive real numbers r_0, b_1, b_2 such that

$$(1.4) \quad b_1 r^\beta \leq \mu(B(z, r) \cap F) \leq b_2 r^\beta$$

for all $z \in F$ and all $r \leq r_0$, where $B(z, r)$ stands for the open ball in \mathbf{R}^d with center z and radius r .

We note that, if D is a bounded Lipschitz domain, then ∂D is a $(d-1)$ -set and the surface measure μ has the property (1.4) for $F = \partial D$ and $\beta = d - 1$. Furthermore if ∂D consists of a finite number of self-similar sets, which satisfies the open set condition, and whose similarity dimensions are β , then ∂D is a β -set such that the β -dimensional Hausdorff measure \mathcal{H}^β restricted to ∂D has the property (1.4) for $F = \partial D$ (cf. [Hu]).

Let $0 < \alpha \leq 1$ and F be a closed set in \mathbf{R}^d . We denote by $\Lambda_\alpha(F \times [0, T])$ the Banach space of all continuous functions f on $F \times [0, T]$ such that $f(\cdot, t)$ is α -Hölder continuous for every $t \in [0, T]$ with norm

$$\|f\|_{\infty, \alpha} = \sup_{X \in F \times [0, T]} |f(X)| + \sup_{x, y \in F, x \neq y, t \in [0, T]} \frac{|f(x, t) - f(y, t)|}{|x - y|^\alpha}.$$

Further let $0 < \alpha, \lambda \leq 1$. We also denote by $\Lambda_{\alpha, \lambda}(F \times [0, T])$ the Banach space of all $f \in \Lambda_\alpha(F \times [0, T])$ such that f is λ -Hölder continuous with respect to the time variable with norm

$$\|f\|_{\infty, \alpha, \lambda} = \|f\|_{\infty, \alpha} + \sup_{x \in F, t, s \in [0, T], t \neq s} \frac{|f(x, t) - f(x, s)|}{|t - s|^\lambda}.$$

We will prove the following lemma in §3.

Lemma 1.1. Let $d-1 \leq \beta < d$ and F be a compact β -set in \mathbf{R}^d satisfying (1.4) and $F \subset B(0, R/2)$. Then there exists a bounded operator \mathcal{E} from $\Lambda_\alpha(F \times [0, T])$ to $\Lambda_\alpha(\mathbf{R}^d)$ with the following properties:

- (i) $\mathcal{E}(f)(\cdot, t)$ is a C^1 -function on $\mathbf{R}^d \setminus F$ for each $t \in [0, T]$, and both of $\mathcal{E}(f)(x, \cdot)$ and $\left(\frac{\partial \mathcal{E}(f)}{\partial x_j}\right)(x, \cdot)$ ($j = 1, \dots, d$) are measurable for each $x \in \mathbf{R}^d$ and for each $x \in \mathbf{R}^d \setminus F$, respectively,
- (ii) $\mathcal{E}(f) = f$ on F and $\text{supp } \mathcal{E}(f)(\cdot, t) \subset B(0, 2R)$ for each $t \in [0, T]$.
- (iii)

$$\left| \frac{\partial \mathcal{E}(f)}{\partial y_i}(y, s) \right| \leq c \|f\|_{\infty, \alpha} \text{dist}(y, \partial D)^{\alpha-1}, \quad \left| \frac{\partial^2 \mathcal{E}(f)}{\partial y_i \partial y_k}(y, s) \right| \leq c \|f\|_{\infty, \alpha} \text{dist}(y, \partial D)^{\alpha-2}$$

for every $(y, s) \in (\mathbf{R}^d \setminus F) \times [0, T]$.

- (iv) If $f \in \Lambda_{\alpha, \lambda}(F \times [0, T])$, then $\mathcal{E}(f) \in \Lambda_{\alpha, \lambda}(\mathbf{R}^d \times [0, T])$.

Using Lemma 1.1 we define, for $f \in \Lambda_\alpha(S_D)$,

$$(1.5) \quad \begin{aligned} \Phi f(X) &= \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(y, s), \nabla_y W(X - Y) \rangle dy \\ &\quad + \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \mathcal{E}(f)(y, s) \Delta_y W(X - Y) dy \end{aligned}$$

for $X = (x, t) \in D \times \mathbf{R}$ and

$$(1.6) \quad \begin{aligned} \Phi f(X) &= - \int_0^T ds \int_D \langle \nabla_y \mathcal{E}(f)(y, s), \nabla_y W(X - Y) \rangle dy \\ &\quad - \int_0^T ds \int_D \mathcal{E}(f)(y, s) \Delta_y W(X - Y) dy \end{aligned}$$

for $X = (x, t) \in (\mathbf{R}^d \setminus \overline{D}) \times \mathbf{R}$.

Furthermore we also define the operator K by

$$(1.7) \quad Kf(Z) = \frac{1}{2} (I_1(Z) + I_2(Z)),$$

where

$$\begin{aligned} I_1(Z) &= \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla \mathcal{E}(f)(y, s), \nabla_y W(Z - Y) \rangle dy \\ &\quad + \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} (\mathcal{E}(f)(Y) - f(Z)) \Delta_y W(Z - Y) dy \\ &\quad + f(Z) \int_{(\mathbf{R}^d \setminus \overline{D}) \times \{0\}} W(Z - Y) dy \end{aligned}$$

and

$$I_2(Z) = - \int_0^T ds \int_D \langle \nabla \mathcal{E}(f)(y, s), \nabla_y W(Z - Y) \rangle dy \\ - \int_0^T ds \int_D (\mathcal{E}(f)(Y) - f(Z)) \Delta_y W(Z - Y) dy - f(Z) \int_{D \times \{0\}} W(Z - Y) dy$$

Under these notations we will prove the following theorem in §3.

Theorem. Assume that D is a bounded domain in \mathbf{R}^d such that ∂D is a β -set. If $0 \leq \beta - (d - 1) < \alpha < 1$ and $f \in \Lambda_{\alpha, \alpha/2}(S_D)$, then, for each $Z \in \partial D \times [0, T]$,

$$(1.8) \quad \lim_{X \rightarrow Z, X \in D \times (0, T)} \Phi f(X) = Kf(Z) + \frac{1}{2}f(Z)$$

and

$$(1.9) \quad \lim_{X \rightarrow Z, X \in (\mathbf{R}^d \setminus \overline{D}) \times (0, T)} \Phi f(X) = Kf(Z) - \frac{1}{2}f(Z).$$

Thus we see that our double layer heat potentials have the same boundary behavior as the usual ones for a bounded cylinder with smooth lateral boundary.

Remark. In this paper we shall treat the double layer heat potentials of Hölder continuous functions on S_D . But under a similar consideration we can also the double layer heat potentials of functions in a Besov space on S_D and prove that they have the parabolically non-tangential limit at a.e. $Z \in S_D$.

2. Properties of W

In this section we recall and study properties of the function W . To do so, we use the parabolic metric δ defined by

$$\delta(X, Y) = (|x - y|^2 + |t - s|)^{1/2} \quad \text{for } X = (x, t) \text{ and } Y = (y, s).$$

- Lemma 2.1.** (i) $W(X) \leq c\delta(X, 0)^{-d}$,
(ii) $|\nabla_x W(X)| \leq c\delta(X, 0)^{-d-1}$ if $X \neq 0$,
(iii) $|\frac{\partial^2}{\partial x_i \partial x_j} W(X)| \leq c\delta(X, 0)^{-d-2}$, $|\frac{\partial}{\partial t} W(X)| \leq c\delta(X, 0)^{-d-2}$ if $X \neq 0$,
(iv) $|\frac{\partial^3}{\partial x_i \partial x_j \partial x_k} W(X)| \leq c\delta(X, 0)^{-d-3}$, $|\frac{\partial^2}{\partial x_j \partial t} W(X)| \leq c\delta(X, 0)^{-d-3}$ if $X \neq 0$,
(v) $|W(X - Y) - W(Z - Y)| \leq c\delta(X, Z)^\epsilon \{\delta(X, Y)^{-d-\epsilon} + \delta(Z, Y)^{-d-\epsilon}\}$
if $0 \leq \epsilon \leq 1$ and $X \neq Y$, $Z \neq Y$,
(vi) $|\nabla_y W(X - Y) - \nabla_y W(Z - Y)| \leq c\delta(X, Z)^\epsilon \{\delta(X, Y)^{-d-1-\epsilon} + \delta(Z, Y)^{-d-1-\epsilon}\}$
if $0 \leq \epsilon \leq 1$ and $X \neq Y$, $Z \neq Y$.

Proof. The assertions (i), (ii), (iii) and (iv) are well known (cf. [B2, p.5]). The assertions (v) and (vi) will be shown by the same method as in the proof of Lemma 2.3 in [W2]. \square

Let D_0 be a bounded piecewise smooth domain in \mathbf{R}^d and u, v be smooth functions on $\overline{D_0} \times [0, \rho]$. Using the divergence theorem, we obtain

$$(2.1) \quad \begin{aligned} & \int_0^\rho \int_{D_0} (uL^*v - vLu) dx dt \\ &= \int_0^\rho dt \int_{\partial D_0} \langle u \nabla_x v - v \nabla_x u, n_x \rangle d\sigma(x) - \int_{D_0 \times \{t=0\}} uv dx + \int_{D_0 \times \{t=\rho\}} uv dx, \end{aligned}$$

where

$$L = \Delta - \frac{\partial}{\partial t} \quad \text{and} \quad L^* = \Delta + \frac{\partial}{\partial t}.$$

If $Lu = L^*v = 0$ in $D_0 \times (0, \rho)$, then (2.1) implies

$$(2.2) \quad \int_0^\rho dt \int_{\partial D_0} \langle u \nabla_x v - v \nabla_x u, n_x \rangle d\sigma(x) - \int_{D_0 \times \{t=0\}} uv dx + \int_{D_0 \times \{t=\rho\}} uv dx = 0.$$

Let $X = (x, t)$ ($0 \leq t \leq T$) be an exterior point of $D_0 \times (0, T)$. Then, setting $u = 1$ and $v(Y) = W(X - Y)$ and noting that $W(X - Y) = 0$ for $Y = (y, T)$, we deduce from (2.2)

$$(2.3) \quad \int_0^T ds \int_{\partial D_0} \langle \nabla_y W(X - Y), n_y \rangle d\sigma(y) - \int_{D_0 \times \{s=0\}} W(X - Y) dy = 0.$$

Hereafter we assume that D is a bounded domain in \mathbf{R}^d such that ∂D is a β -set satisfying $\overline{D} \subset B(O, R/2)$.

Let us use the Whitney decomposition to approximate D and $\mathbf{R}^d \setminus \overline{D}$ (cf. [S, p.167]). Let $\mathcal{V}(D)$ be the Whitney decomposition of D and define

$$A_n = \bigcup_{k=k_0}^n \bigcup_{Q \in \mathcal{V}_k(D)} Q,$$

where $\mathcal{V}_k(D) = \{Q \in \mathcal{V}(D); Q \text{ is a } k\text{-cube}\}$ and k_0 is the smallest integer k such that $\mathcal{V}_k(D) \neq \emptyset$.

Similarly we also define

$$B_n = \left(\bigcup_{k=-\infty}^n \bigcup_{Q \in \mathcal{V}_k(\mathbf{R}^d \setminus \overline{D})} Q \right).$$

Then we have the following lemma.

Lemma 2.2. Set

$$g_n(X) = \int_0^T \int_{A_n} \Delta_y W(X - Y) dy \quad \text{and} \quad h_n(X) = \int_0^T \int_{B_n} \Delta_y W(X - Y) dy.$$

Then $\lim_{n \rightarrow \infty} g_n(X)$ and $\lim_{n \rightarrow \infty} h_n(X)$ exist on $\mathbf{R}^d \times [0, T]$ and for $X \in \mathbf{R}^d \times (0, T]$

$$\lim_{n \rightarrow \infty} g_n(X) = \int_{D \times \{0\}} W(X - Y) dy - \chi_D(X)$$

and

$$\lim_{n \rightarrow \infty} h_n(X) = \int_{(\mathbf{R}^d \setminus \overline{D}) \times \{0\}} W(X - Y) dy - \chi_{\mathbf{R}^d \setminus \overline{D}}(X)$$

Proof. Let $X = (x, t) \in \mathbf{R}^d \times (0, T]$ and $t > \rho > 0$. Applying (2.2) to $A_n \times (0, \rho)$, we have

$$\begin{aligned} & \int_0^\rho ds \int_{\partial A_n} \langle \nabla_y W(X - Y), n_y \rangle d\sigma(y) - \int_{A_n \times \{0\}} W(X - Y) dy \\ & + \int_{A_n \times \{\rho\}} W(X - Y) dy = 0. \end{aligned}$$

Using the divergence theorem for A_n in \mathbf{R}^d , we have

$$\begin{aligned} & \int_0^\rho ds \int_{A_n} \Delta_y W(X - Y) dy - \int_{A_n \times \{0\}} W(X - Y) dy \\ & + \int_{A_n \times \{\rho\}} W(X - Y) dy = 0. \end{aligned}$$

As $\rho \rightarrow t$ and $n \rightarrow \infty$, we obtain,

$$\lim_{n \rightarrow \infty} g_n(X) = \int_{D \times \{0\}} W(X - Y) dy - \chi_D(X).$$

On the other hand $g_n(X) = 0$ for $t = 0$. Hence $\lim_{n \rightarrow \infty} g_n(X)$ exists for each $X \in \mathbf{R}^d \times [0, T]$.

Similarly we can also prove the conclusion for h_n .

□

3. Double layer heat potentials

In this section we first prove Lemma 1.1 in §1.

Proof of Lemma 1.1 We use the extension operator \mathcal{E}_0 in [S, p.172] and choose a C^∞ -function ϕ_0 such that

$$\phi_0 = 1 \quad \text{on } B(0, R), \quad \text{supp } \phi_0 \subset B(0, 2R) \quad \text{and } 0 \leq \phi_0 \leq 1.$$

We define

$$\mathcal{E}(f)(x, t) = \mathcal{E}_0(f(\cdot, t))(x)\phi_0(x)$$

for $f \in \Lambda_\alpha(F)$ and $(x, t) \in (\mathbf{R}^d \setminus F) \times [0, T]$ and

$$\mathcal{E}(f)(x, t) = f(x, t) \quad \text{on } (x, t) \in F \times [0, T].$$

Then properties (i), (ii), (iii) follow from the definition and (13) on p.174 in [S]. Since the operator \mathcal{E}_0 is linear, positive and maps the constant function 1 to 1, (iv) is also valid. \square

In [W1] we gave the following lemma.

Lemma A. *Let δ, k be non-negative numbers satisfying $d - \beta > \delta$ and $d - \delta - k > 0$. Then*

$$\int_{B(z, r)} \text{dist}(y, \partial D)^{-\delta} |y - z|^{-k} dy \leq cr^{d-\delta-k}$$

for every $z \in \partial D$ and $r > 0$.

We next show that the double layer heat potential defined by (1.5) and (1.6) converges.

Lemma 3.1. *Let $0 \leq \beta - (d - 1) < \alpha < 1$ and $f \in \Lambda_\alpha(S_D)$. Then Φf is caloric in $(\mathbf{R}^d \setminus \partial D) \times \mathbf{R}$.*

Proof. Set, for $X = (x, t) \in D \times \mathbf{R}$,

$$(3.1) \quad J_1(X) = \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(y, s), \nabla_y W(X - Y) \rangle dy$$

and let $X_0 = (x_0, t_0) \in D$. Choose $\rho > 0$ satisfying $\overline{B(x_0, 2\rho)} \subset D$. If $X = (x, t) \in B(x_0, \rho) \times \mathbf{R}$, then we deduce from Lemmas 2.1 and 1.1 and Lemma A

$$|J_1(X)| \leq \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \text{dist}(y, \partial D)^{\alpha-1} \delta(X, Y)^{-1-d} dy \leq c_1 \rho^{-1-d} \|f\|_{\infty, \alpha},$$

whence J_1 converges locally uniformly in D . We denote by g_1 the integrand of the right-hand side on (3.1). Since

$$|\nabla_y \frac{\partial^2}{\partial x_i \partial x_j} W(X - Y)| \leq c_2 \delta(X, Y)^{-d-3} \quad \text{and} \quad |\nabla_y \frac{\partial}{\partial t} W(X - Y)| \leq c_3 \delta(X, Y)^{-d-3},$$

we see that the integral of Lg_1 over $(\mathbf{R}^d \setminus \overline{D}) \times [0, T]$ also converges locally uniformly on D . Therefore J_1 satisfies the heat equation in $D \times \mathbf{R}$.

Next, set

$$J_2(X) = \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \mathcal{E}(f)(y, s) \Delta W(X - Y) dy.$$

Using Lemma 1.1, (iii), we can show by the above method that J_2 also converges locally uniformly in D and satisfies the heat equation. Thus we conclude that $\Phi f = J_1 + J_2$ has the same properties in $D \times \mathbf{R}$. We can show that Φf also has the same properties in $(\mathbf{R}^d \setminus \overline{D}) \times \mathbf{R}$. \square

Using Lemma 2.1, (iv), (v) and Lemma A, we can prove the following lemma by a similar method to that in the proof of [W1, Lemma 3.3].

Lemma 3.2. *Let $0 \leq \beta - (d - 1) < \alpha < 1$ and $f \in \Lambda_{\alpha, \alpha/2}(S_D)$. Then both of the function J_1 defined by (3.1) and the function J_3 defined by*

$$J_3(X) = \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} (\mathcal{E}(f)(Y) - \mathcal{E}(f)(X)) \Delta W(X - Y) dy$$

are continuous on $\mathbf{R}^d \times [0, T]$. Furthermore the function J'_1 (resp. J'_3) obtained by replacing $\mathbf{R}^d \setminus \overline{D}$ with D in the definition of J_1 (resp. J_3) is also continuous on $\mathbf{R}^d \times [0, T]$.

Lemma 3.3. *Let $0 \leq \beta - (d - 1) < \alpha < 1$ and $g \in \Lambda_{\alpha, \alpha/2}(\mathbf{R}^d \times [0, T])$ such that $g(\cdot, t) \in C^1(\mathbf{R}^d)$, $\text{supp } g(\cdot, s) \subset B(0, r_0)$ for every $t \in [0, T]$ and $\frac{\partial g}{\partial x_j}(x, \cdot)$ is bounded for every $x \in \mathbf{R}^d$. Let $X = (x, t) \in \mathbf{R}^d \times (0, T]$ and set, for $0 < \rho \leq T$,*

$$\begin{aligned} A_\rho g(X) &= \int_0^\rho ds \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla g(Y), \nabla_y W(X - Y) \rangle dy \\ &\quad + \int_0^\rho ds \int_{\mathbf{R}^d \setminus \overline{D}} (g(Y) - g(X)) \Delta_y W(X - Y) dy \\ &\quad + g(X) \int_{(\mathbf{R}^d \setminus \overline{D}) \times \{0\}} W(X - Y) dy \end{aligned}$$

and

$$\begin{aligned} B_\rho g(X) &= - \int_0^\rho ds \int_D \langle \nabla g(Y), \nabla_y W(X - Y) \rangle dy \\ &\quad - \int_0^\rho ds \int_D (g(Y) - g(X)) \Delta_y W(X - Y) dy \\ &\quad - g(X) \int_{D \times \{0\}} W(X - Y) dy. \end{aligned}$$

Then

$$(3.2) \quad A_T g(X) = B_T g(X) + g(X) \text{ for } X \in \mathbf{R}^d \times (0, T]$$

Proof. To simplify the notation, we use $A_\rho(x)$ and $B_\rho(X)$ instead of $A_\rho g(X)$ and $B_\rho g(X)$, respectively. We first show (3.2) in case $D = D_0$ is a bounded piecewise smooth domain. Let $X = (x, t)$ and set, for $0 < \rho < t$,

$$I_\rho(X) = - \int_0^\rho ds \int_{\partial D_0} g(Y) \langle \nabla_y W(X - Y), n_y \rangle d\sigma(y).$$

The Green formula for D_0 yields

$$(3.3) \quad \begin{aligned} I_\rho(X) = & - \int_0^\rho ds \int_{D_0} \langle \nabla g(Y), \nabla_y W(X - Y) \rangle dy \\ & - \int_0^\rho ds \int_{D_0} (g(Y) - g(X)) \Delta_y W(X - Y) dy \\ & - g(X) \int_0^\rho ds \int_{D_0} \Delta_y W(X - Y) dy \end{aligned}$$

From (2.2) we deduce

$$\begin{aligned} & \int_0^\rho ds \int_{D_0} \Delta_y W(X - Y) dy \\ & = \int_{D_0 \times \{0\}} W(X - Y) dy - \int_{D_0 \times \{\rho\}} W(X - Y) dy, \end{aligned}$$

whence

$$\int_0^t ds \int_{D_0} \Delta_y W(X - Y) dy = \int_{D_0 \times \{0\}} W(X - Y) dy - \chi_{D_0}(x).$$

This and (3.3) imply

$$(3.4) \quad I_t(X) = B_t(X) + g(X) \chi_{D_0}(x) \text{ for } X \in (\mathbf{R}^d \setminus \partial D_0) \times (0, T].$$

Similarly, using the Green formula for $B(0, r) \setminus \overline{D_0}$ and $r \rightarrow \infty$, we obtain

$$I_t(X) = A_t(X) - g(X) \chi_{\mathbf{R}^d \setminus \overline{D_0}}(x)$$

for $X \in (\mathbf{R}^d \setminus \partial D_0) \times (0, T]$. This and (3.4) lead to

$$A_t(X) = B_t(X) + g(X) \text{ for } X \in (\mathbf{R}^d \setminus \partial D_0) \times (0, T].$$

Noting that $A_t(X) = A_T(X)$ and $B_t(X) = B_T(X)$, we obtain (3.2) for $X \in (\mathbf{R}^d \setminus \partial D) \times (0, T]$. Since A_T and B_T are continuous on $\mathbf{R}^d \times (0, T]$ by Lemma 3.2, (3.2) holds for a bounded piecewise smooth domain $D = D_0$.

We next show (3.2) for a bounded domain such that ∂D is a β -set. We use (3.2) for $D_0 = A_n$. Since

$$\begin{aligned} \int_0^T \int_{\mathbf{R}^d} |\nabla g(Y)| |\nabla_y W(X - Y)| dy ds &< \infty, \\ \int_0^T ds \int_{\mathbf{R}^d} |g(Y) - g(X)| |\Delta_y W(X - Y)| dy &< \infty \end{aligned}$$

and

$$\int_{\mathbf{R}^d \times \{0\}} W(X - Y) dy < \infty,$$

we see that (3.2) holds for the domain D as $n \rightarrow \infty$. \square

Lemma 3.4. *Let $0 \leq \beta - (d - 1) < \alpha$ and $f \in \Lambda_{\alpha, \alpha/2}(S_D)$. Then (3.2) holds for $g = \mathcal{E}(f)$.*

Sketch of Proof. Let $f \in \Lambda_{\alpha, \alpha/2}(S_D)$ and $\{v_m\}$ be a mollifier on \mathbf{R}^d such that $\text{supp } v_m \subset B(0, 1/m)$. We define, for $Y = (y, s) \in \mathbf{R}^d \times [0, T]$,

$$g_m(Y) = (\mathcal{E}(f)(\cdot, s) * v_m)(y).$$

Lemma 3.3 yields

$$A_T g_m(X) = B_T g_m(X) + g_m(X) \text{ for } X \in \mathbf{R}^d \times (0, T].$$

Using $g_m(X) \rightarrow \mathcal{E}(f)(X)$ uniformly as $m \rightarrow \infty$ and Lemmas A, 1.1 and 2.1, we can show that

$$A_T g_m(X) \rightarrow A_T \mathcal{E}(f)(X)$$

and

$$B_T g_m(X) \rightarrow B_T \mathcal{E}(f)(X)$$

for $X \in \mathbf{R}^d \times [0, T]$ as $m \rightarrow \infty$. \square

We can also show the following lemma.

Lemma 3.5. *Let $0 \leq \beta - (d - 1) < \alpha < 1$. Then the operator K defined by (1.7) is a bounded operator from $\Lambda_{\alpha, \alpha/2}(S_D)$ to $\Lambda_{\alpha, \alpha/2}(S_D)$.*

Let us prove our theorem.

Proof of Theorem. Let $X \in D \times (0, T]$. Using Lemma 2.2, we have $\Phi f(X) = A_T f(X)$. Since $A_T f$ is continuous on $\mathbb{R}^d \times (0, T]$ by Lemma 3.2, we have

$$\lim_{X \rightarrow Z, X \in D \times (0, T)} \Phi f(X) = A_T f(Z).$$

On the other hand Lemma 3.4 yields

$$Kf(Z) = \frac{1}{2}(A_T f(Z) + B_T f(Z)) = A_T f(Z) - \frac{1}{2}f(Z).$$

Therefore we have (1.8). Similarly we can show (1.9). □

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